Optimal Reputation Systems for Platforms*

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Abstract

We study a continuous-time platform design problem. Each agent's output is an imperfect signal of his underlying choice of effort, and each agent's utility from remaining on the platform is endogenous to the output of the other agents. Monetary transfers are assumed infeasible. Incentives can be provided only through two potential channels: termination due to poor performance (the stick) and rewarding good performance by allowing a spell of shirking (the carrot). We derive the steady-state distribution of continuation utilities of agents on the platform and show that both the stick and the carrot are used to provide incentives under an optimal design. Moreover, the optimal platform design may be implemented by associating continuation utilities with a reputation system that tracks an agent's performance over time.

Keywords: platforms, continuous time, sticky Brownian motion, stationary distribution, contracts with shirking

JEL Classification Numbers: C73, D86, M52

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1 Introduction

Numerous interactions in the modern economy are conducted on platforms that are controlled by a mediator or principal who is able to track the outcome history of individual agents. Examples include ride and home sharing firms, auction sites, social networks, and dating services among others. Incentives in these settings are seldom provided through explicit monetary channels. Rather, the platform host typically tracks the history of each agent's transactions and threatens to punish poor performance with expulsion.

In this paper we analyze a continuous-time platform design problem where the principal must manage incentives for a continuum of agents who may make costly contributions to each other's payoffs. Specifically, the designer maintains a reputation system tied to agents' performance, whereby agents who hit the bottom of the reputation scale are expelled from the platform. We also show that agents who hit the top of the scale *must* be allowed to shirk for a nonnegligible amount of time; at the top of the scale, an agent's situation can improve no further, and thus it is impossible to incentivize effort. Because expulsion and shirking are both inefficient, each agent's reputation is linked to his stochastic output with the minimal sensitivity consistent with incentive compatibility.

Given the law of motion of agents' continuation values, we employ novel methods to derive the steady-state of such a platform, where the density of agents at each level of reputation (or continuation utility) remains constant over time. Using this steady-state, we frame the principal's problem which can involve various objective functions. The characteristics of the steady-state distribution are governed by two parameters: the highest attainable level of continuation utility on the platform, w^*, and the level at which new agents who enter the platform are inserted, w^0. The steady-state distribution also features several intriguing features. First, by definition, $w = 0$ is a *resetting boundary* in the sense that the flow of agents being expelled from the platform at that point must equal the flow of new agents entering at w^0. Second, w^* is a *sticky boundary* or *slow-reflecting barrier* in the sense that an agent reaching this level spends a positive expected measure of time there. Moreover, an agent exerts low effort if and only if his continuation utility is exactly w^*. Together these observations imply that the steady-state distribution possesses a positive mass of agents at the top of the scale who are rewarded by being allowed to exert low effort. Importantly, since we are studying a platform where agents' efforts have externalities, these reward periods affect all agents (and their incentives) by reducing their flow benefits from remaining on the platform.

For the platform to be sustainable, agents must earn a positive flow payoff from being on the platform. However, agents are rewarded by being allowed to shirk, and when a larger fraction of agents is shirking, all agents obtain a reduced flow payoff from the platform. Hence, the principal's ability to reward agents is self-limiting. For most of the paper, we assume that agents get a fixed, positive flow payoff from the platform as long as the fraction of agents working is above some threshold. This results in a *feasible set* of design parameters, w^0 and w^*. We characterize the feasible set and show that it lies below some single-peaked curve $\bar{w}^0(w^*)$ in (w^*, w^0)-space.

Next, we investigate the principal's platform design problem; namely, we allow her to choose w^* and w^0 subject to feasibility constraints. Depending on the institutional setting, the principal could have a variety of objective functions. For instance, a principal may want to maximize the per-capita output of agents, which would involve maximizing the fraction of agents working. We show that this objective leads to a vanishingly small platform: agents are promised an arbitrarily small continuation value with virtually no hope of reaching the shirking state before being kicked off the platform. Alternatively, a profit-maximizing principal who could charge an entry fee to the platform (e.g., a dating site) would want to set w^0 as high as possible. Under certain conditions, the entire feasible set lies below the 45-degree line in (w^*, w^0)-space, and thus it is possible that the entry-fee maximizing starting value is strictly lower than the maximum continuation value – despite that agents are very likely to reach higher continuation values than w^0, the principal cannot possibly promise them a higher starting value without violating feasibility, since higher starting values imply a larger fraction of agents shirking in a steady state. On the other hand, a principal who generates revenue primarily through advertising (e.g., a social network) would want to maximize the size of the platform. In this case, the principal's objective is increasing in both w^* and w^0, and an optimal platform lies on the northeastern frontier of the feasible set. Whatever her objective, we note that by identifying continuation utilities with reputation scores, the principal's problem can be thought of as the design of an optimal reputation system.

Literature

Our paper contributes to several growing strands of research. The first, on the use of ratings and reputation for incentive compatibility, began with Holmström (1999) and includes–among others–a recent working paper by Horner and Lambert (2016). These authors investigate the use of reputation as a means for eliciting the rated agent's cooperation in a setting where the principal has imperfect control over compensation. In order to maintain a high reputation–and thereby a high continuation payoff–agents are required to produce a stream of signals reflective of high effort. We use reputation in much the same way; in our setting, an agent is allowed a shirking bonus when his reputation is at the top of the scale, and he is fired when his reputation hits bottom. At intermediate levels the agent receives a constant flow payoff so that incentives are provided entirely by the payoffs associated with the reputational extremes. Control over the distance between the top and bottom of the scale, therefore, gives the platform designer in our model some power to influence continuation payoffs and to provide incentives for maintaining high effort. This implementation is, to the best of our knowledge, novel in the optimal contracting setting where we apply it.

Another strand of related research is the continuous time optimal contracting literature in corporate finance. The pioneering article is DeMarzo and Sannikov (2006) (hereafter DS), which was followed by a number of related works including Sannikov (2008), Zhu (2013), and Grochulski and Zhang (2016). Each of these papers investigate variations of the DS baseline model which we, too, adapt to our setting. Specifically, each of them considers a single agent who may take some action to produce output or to benefit himself, and solves for the optimal path of continuation values in order to maximize output. DS implement their optimal contract by way of basic financial instruments, while the others abstract from implementation considerations. Like Zhu and Grochulski and Zhang, we find that shirking is necessary at some points (at the highest level of continuation utility in our setting), where reputation is temporarily insensitive to performance. Shirking manifests technically as sticky (or slowly reflecting) Brownian Motion, formalized by Harrison and Lemoine (1981).

There is also a literature on Folk theorems in continuous time with imperfect monitoring associated with Brownian noise, including Sannikov (2007), Peski and Wiseman (2015), and Bernard and Frei (2016). These papers derive characterizations of the sets of achievable payoffs in continuous-time stochastic games such as ours; most closely related is Sannikov (2007) which uses methods similar to DS and Zhu (2013) that we also employ.[1]

Finally, our paper contributes to the organizational economics literature. Specifically, because each agent in our model receives a gross expected payoff equal to the average output of the platform, our setting resembles a dynamic *partnership* in which output is divided equally among the members of the organization as in Farrell and Scotchmer (1988) and Levin and Tadelis (2005). This partnership aspect, along with the steady-state analysis is what distinguishes our model from other recent work on dynamic relational contracts such as Andrews and Barron (2016).

The model is introduced in the next section along with an overview of the method. In Section 3 we solve for the steady-state distribution of continuation utilities (i.e., reputations) on the platform for arbitrary values of the policy instruments. Section 4 characterizes the feasible set. In Section 5 we discuss the principal's platform design problem for various

[1]See also the seminal discrete-time models of Abreu et al. (1990) and Green and Porter (1984).

possible objectives. We outline direction for future work in Section 6.

2 Setup

Time is assumed continuous over an infinite horizon. At each moment there is a positive measure of massless agents present on a *platform*. Agents on the platform are indexed by a continuous variable i. Each agent is risk neutral and discounts the future at rate r. An agent who *exits* the platform receives a payoff of 0 from that point forward. A flow of new agents $\psi > 0$ enter the platform at each instant.

While on the platform, each agent receives a flow utility u which will, in equilibrium, be generated by the actions of all the agents on the platform itself. For now, we assume u is a constant, exogenously determined flow payoff. At each instant, each agent i chooses an effort level $e^i \in \{H, L\}$; thus each agent i chooses a stream of effort levels, which is a stochastic process $(e^i_t)_{t\geq 0}$. The flow cost of effort is $c(e^i_t)$, defined by $c(H) = c > 0$ and $c(L) = 0$. Thus, high effort has a flow cost $c > 0$ and low effort has no cost. An agent's effort generates a stream of output given by a Brownian diffusion[2]

$$dX^i_t = (\mu_{e^i_t})dt + dB^i_t. \tag{2.1}$$

The principal (i.e., the platform operator) observes each agent's stream of output, but not his effort choice. So that high effort is efficient and low effort is not, we assume that $\mu_H - c > 0 > \mu_L$. Since agents with the same continuation utilities are essentially identical, we suppress the index i whenever doing so does not create confusion. Below we also speak of *the agent* with the understanding that we are focussing on a single arbitrary participant on the platform. A contract in this context specifies: (i) a fixed flow utility u, (ii) a termination time τ and (iii) a recommended effort process e_t. Here we consider permanent expulsion at τ. Also, an agent is free to leave the platform at any point, but may not reenter.

In order to implement a contract, at each instant, the principal assigns each agent a score or *reputation*, $s^i_t \in [0, s^*]$. Incoming agents begin with reputation s^0, and reputations evolve according to their observed output. The distribution of agents at each instant thus is characterized by a population distribution over $[0, s^*]$. Furthermore, as the agent is motivated solely by the evolution of his continuation payoff, the designer may simply set the reputation process S_t for each agent equal to his continuation payoff process W_t, thus controlling directly the agents' payoffs. In particular, an agent is removed from the platform when his reputation

[2] Although we follow the convention of calling dX^i_t the *output* stream of agent i, it should b more generally interpreted as just a stream of signals on agent i's unobservable effort process, which need not be directly related to his contribution to the output of the platform.

$s_t = w_t = 0$, new agents are granted a continuation payoff of $w^0 = s^0$, and agents are never allowed a continuation payoff above the maximum reputation level, so that $w^* = s^*$ is the maximum continuation payoff. In this way, we ultimately will allow the principal to directly choose w^0 and w^* as a part of her platform design problem.

From standard results in continuous-time contracting,[3] the following are known:

- While the agent remains on the platform (i.e., $t \leq \tau$), there exists a process β_t representing the sensitivity of the agent's continuation value to output:

$$dW_t = rW_t - (u - c(e_t))dt + \beta_t(dX_t - \mu_{e_t} dt).$$

- The contract is incentive compatible if and only if for all $t \leq \tau$ and $W_t \geq 0$, $e_t = H$ implies $\beta_t \geq \lambda$ and $e_t = L$ implies $\beta_t \leq \lambda$, where $\lambda := \frac{c}{\mu_H - \mu_L}$.

So long as the platform designer's objective is increasing in effort, terminating an agent is inefficient (i.e., on-path agents are terminated due to bad luck, not because they were shirking at the moment). Therefore, the principal wishes to minimize volatility, and thus it is optimal to set $\beta_t = \lambda$ to induce working and $\beta_t = 0$ to induce shirking. Hence, when the agent works, his continuation value evolves as

$$\begin{aligned} dW_t &= rW_t dt - (u - c)dt + \lambda(dX_t - \mu_H dt) \\ &= (rW_t - (u - c))dt + \lambda dB_t. \end{aligned} \tag{2.2}$$

When the agent shirks, his continuation value evolves (deterministically) as

$$dW_t = rW_t dt - u dt. \tag{2.3}$$

The best outcome for the agent would be to shirk at all times and remain on the platform forever, so we have $W_t \leq u/r$. Through (2.3), this implies that continuation value has downward drift while the agent works. For later convenience, we define $\rho(w) := rw - u$ so that $\rho(w^*)$ is the downward drift at w^*. Now suppose that w^* is an upper bound on the agent's continuation value. The law of motion (2.2) while the agent works guarantees that the agent's continuation value will reach w^* in finite time almost surely, and shirking must occur at this point. Formally, the process W_t that results belongs to a class of diffusions known as *Sticky Brownian Motion*.[4]

[3]See DeMarzo and Sannikov (2006) Lemmas 2 and 3 or Zhu (2013) Lemmas 3.1 and 3.2.
[4]See Harrison and Lemoine (1981) and Zhu (2013).

2.1 Overview of Method

Now that the model has been presented, the method we use to solve our platform design problem is outlined in four steps:

1. *Determine agents' law of motion:* fixing contract terms (u, w^0, w^*) exogenously such that incentive compatibility holds, characterize the evolution of the state variable for each agent as a stochastic process. In the current paper, the contract specifies a firing rule as a function of the agent's output process, which allows shirking at certain times. The relevant state variable is the agent's continuation value, w.

2. *Find steady-state distribution:* determine a stationary distribution for the state variable, using the law of motion from above along with the conditions arising from the contract specifics, (u, w^0, w^*).

3. *Endogenize flow payoffs:* since agents on a platform exert externalities on one another, the flow payoffs they earn u must be consistent with the steady-state distribution of agents and their actions. This requirement determines a feasible set of platform design parameters, namely w^0 and w^*, over which the platform designer can optimize.

4. *Optimize platform design parameters:* for a given objective function for the platform designer, determine the optimal values of w^0 and w^* from the feasible set derived in step 3.

3 The Steady State

A steady state corresponds to a stationary distribution of agents at all possible levels of continuation payoff, as poorly performing agents drop off the platform, new agents arrive, and as the continuation value of all agents within the system are changing in response to their Brownian output.

The formal continuous-time derivation of the stationary distribution is given in the appendix. To illustrate the derivation, we discretize the distribution of agents as well as the continuation value process. Suppose that the maximum continuation value is w^* and that new agents begin with a continuation value of $w^0 \in (0, w^*)$. Mathematically, the steady state in our setting is equivalent to the stationary distribution of a process W defined on an interval $[0, w^*]$ as follows:

- When $W_t = 0$, it immediately resets to w^0

- For $W_t \in (0, w^*)$, it evolves as (2.2)

- For $W_t = w^*$, it evolves as (2.3).

We begin by characterizing the stationary distribution away from the special points $0, w^0$ and w^*. Suppose that at a particular continuation value $w \in (0, w^0) \cup (w^0, w^*)$, a mass $f(w)dw$ of agents resides. The law of motion (2.2) for an individual agent's continuation value can be approximated by a random walk as follows: starting from w, it moves up or down a step of size $dw = \lambda\sqrt{dt}$, where upward steps occur with probability

$$q(w; dt) := .5 \left(1 + \frac{rw - (u - c)}{\lambda}\sqrt{dt}\right).$$

In a steady state, the measure of agents exiting from each w must equal the measure of agents entering. Since all agents starting from w must exit (up or down), we have

$$f(w)dw = \underbrace{f(w - dw)dwq(w - dw; dt)}_{\text{agents moving up from } w - dw} + \underbrace{f(w + dw)dw(1 - q(w + dw; dt))}_{\text{agents moving down from } w + dw}.$$

By performing a second-order Taylor expansion of the right side with respect to w, dropping terms of order dt and higher and simplifying, we obtain the standard Kolmogorov forward equation

$$rf(w) + (rw - (u - c))f'(w) = \frac{\lambda^2}{2} f''(w). \tag{3.1}$$

The general solution to the ODE (3.1) can be written in closed form and has two constants of integration:

$$f(w) = e^{\gamma(w)^2} \left(C_1 + C_2 \text{erf}\left\{\gamma(w)\right\}\right), \tag{3.2}$$

where $\gamma(w) := \frac{rw-(u-c)}{\lambda\sqrt{r}}$ and $\text{erf}\left\{x\right\} := \frac{2}{\sqrt{u}} \int_0^x e^{-t^2} dt$ is the Gauss error function. Note that (3.2) must hold separately on both the left and right segments of the distribution. Let $f_-(w)$ and its associated constants C_1^-, C_2^- denote the solution to (3.2) on the interval $(0, w^0)$, and likewise define $f_+(w)$, C_1^+ and C_2^+ on the interval (w^0, w^*).

Since the process W is a sticky Brownian motion, there is a positive measure of times t at which $W_t = w^*$. This implies that the stationary distribution involves an atom of mass at exactly w^*. Let ν denote the measure associated with the steady state distribution on $[0, w^*]$, so that the mass at w^* is $\nu\{w^*\}$. For $w \in (0, w^0]$ we have $\nu(dw) = f_-(w)dw$ and for $w \in [w^0, w^*)$ we have $\nu(dw) = f_+(w)dw$. There are five constants in total to be determined $(\nu\{w^*\}, C_1^-, C_2^-, C_1^+, C_2^+)$, and these are pinned down by five boundary conditions stated in Lemma 3.1. We now provide a heuristic derivation of these boundary conditions.

To obtain boundary conditions 1 and 5, we note that in a short interval of time dt, the measure of agents exiting the platform must equal the exogenous inflow ψdt. To approximate the former, note that the measure of agents at dw (i.e., one step above 0) is $f_-(dw)dw = \lambda\sqrt{dt}f_-(0) + \lambda^2 dt f_-'(0)$. Of these, a fraction $1 - q(dw; dt)$ move a downward step, for a total exiting measure of

$$(1 - q(dw; dt))f_-(dw)dw = \frac{\lambda}{2}\sqrt{dt}f_-(0) + dt\left[\frac{\lambda^2}{2}f_-'(0) - (rw - (u - c))f_-(0)\right].$$

Since the above must equal ψdt, it must be that the term of order $\sqrt{dt}$ vanishes; that is, $f_-(0) = 0$, which is condition 1 in Lemma 3.1 below. Matching terms of order dt, we obtain condition 5.

Condition 2 is a standard value-matching condition which results from the continuity and volatility of Brownian motion. Condition 3 equates the outflow of agents at 0 to the inflow of agents at w^0. By the above results, the outflow of agents at 0 is $\frac{\lambda^2}{2}dt f_-'(0)$. The inflow of agents at w^0 is given by the concavity of the kink between densities at w^0 (see Figure 1), which is $f_+(w)dw - \frac{f_-(w-dw)dw + f_+(w+dw)dw}{2} = \frac{\lambda^2}{2}dt\left[f_-'(w^0) - f_+'(w^0)\right]$. Equating the inflow and outflow, we obtain condition 3. Intuitively, this condition says that the faster agents exit at 0, the more pronounced the kink will be at w^0.

Finally, condition 4 relates the (stochastic) upward movement of agents just below w^* to the (deterministic) downward movement of agents at w^* due to sticky reflection. To a first order approximation, the measure of agents just below w^* is $f_+(w^*-)dw$, of which 1/2 move a step upward. The measure of agents at exactly w^* is $\nu\{w^*\}$, all of which move down a distance $-\rho(w^*) > 0$; equivalently, a multiple $\frac{-\rho(w^*)}{dw}$ move down one step on the random walk grid. Equating the upward and downward movements yields $\frac{1}{2}f_+(w^*)dw = \frac{-\rho(w^*)}{dw}\nu\{w^*\}$, which rearranges to condition 4. Since $\rho(w^*)$ is closer to 0 when w^* is closer to u/r, this condition says that higher continuation values are sustained by letting the agent shirk longer, in the sense of slowing down the rate of sticky reflection. In turn, slower reflection results in a larger atom of agents at w^* in a steady state, holding fixed $f(w^*-)$.

Lemma 3.1. *The steady-state distribution of agents is characterized by an atom $\nu\{w^*\}$ and piecewise densities f_- and f_+ of the form (3.2) defined on $(0, w^0]$ and $[w^0, w^*)$, respectively, subject to the following boundary conditions:*

1. $f_-(0+) = 0$.

2. $f_-(w^0) = f_+(w^0)$.

3. $f_-'(0+) = f_-'(w^0) - f_+'(w^0)$.

4. $\frac{\lambda^2}{2}f_+(w^*-) + \rho(w^*)\nu\{w^*\} = 0$.

5. $\frac{\lambda^2}{2} f'_-(0+) = \psi$.

In the appendix, we show that the boundary conditions admit a unique solution, and all of the constants are determined explicitly. These results are summarized in the following proposition.

Proposition 3.1. *In a steady state, the distribution of agents is as follows:*

- $f_-(w) = e^{\gamma(w)^2} \frac{\psi\sqrt{u}}{\lambda\sqrt{r}} \left[\mathrm{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \mathrm{erf}\left\{\gamma(w)\right\}\right]$
- $f_+(w) = e^{\gamma(w)^2} \frac{\psi\sqrt{u}}{\lambda\sqrt{r}} \left[\mathrm{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \mathrm{erf}\left\{\gamma(w^0)\right\}\right]$
- $\nu\{w^*\} = \frac{\lambda^2 f_+(w^*)}{2(u-rw^*)}$

This fully characterizes the distribution of continuation utilities on the platform in a steady state as a function of the highest achievable continuation utility, w^* and the level at which new agents are inserted w^0. In words, the distribution consists of a density function composed of two convex segments (with a kink where they meet at w^0) as well as a mass point at the top of the support, w^*; Figure 1 illustrates. The question is what values for w^0 and w^* the principal can and should set.

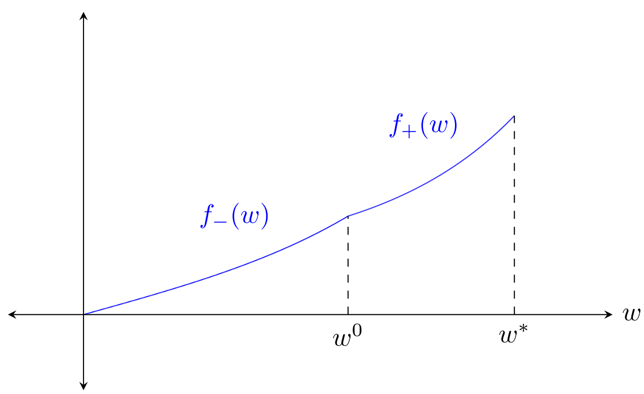


Figure 1: Steady state distribution of agents (omitting the atom at w^*).

4 The Feasible Set

Prior to this section it has sufficed to interpret u as an exogenous flow payoff that the principal can promise the agents on the platform. In this section, we impose the constraint that each agent's flow payoff is a function of the efforts of other agents on the platform. Since the latter depends on the design parameters and u itself, u is a fixed point. We assume that u is a nondecreasing function of the *fraction* of agents working on the platform.

This captures either of two distinct kinds of interactions on real-world platforms: (i) agents randomly match with other agents for short interactions, or (ii) agents collectively produce some public good that gets distributed evenly among them.

Define the measure of agents active (i.e., working) in a steady state by

$$\alpha(u, w^0, w^*) := \int_0^{w^0} f_-(w)\, dw + \int_{w^0}^{w^*} f_+(w)\, dw,$$

and (with some abuse of notation) define the measure of non-active agents at the top of the distribution by

$$\nu(u, w^0, w^*) := \nu\{w^*\}.$$

Denote the fraction of agents working by

$$Q(u, w^0, w^*) := \frac{\alpha(u, w^0, w^*)}{\alpha(u, w^0, w^*) + \nu(u, w^0, w^*)}. \tag{4.1}$$

The endogenous flow utility produced when $Q = Q(u, w^0, w^*)$ are working is $U(Q)$, where $U : [0, 1] \to \mathbb{R}_+$ is a nondecreasing function which is not identically zero.

In order to define feasibility, we state two conditions:

$$0 < w^0 \leq w^* \leq u/r \tag{4.2}$$

$$u = U(Q(u, w^0, w^*)). \tag{4.3}$$

Definition 4.1. *A platform* (u, w^0, w^*) *is feasible if* (4.2) *and* (4.3) *are satisfied.*

The key part of Definition 4.1 is the fixed point condition $u = U(Q(u, w^0, w^*))$. This condition says that the flow payoff that each agent receives (which the principal exogenously "promises") must equal the flow payoff generated by the platform. Condition (4.2) says that u must be non-negative or agents would be better off leaving the platform (individual rationality). Moreover, $w^* \leq \frac{u}{r}$ because the highest level of continuation utility can at most equal the perpetuity value of shirking forever (i.e., when the process is infinitely sticky, w^* becomes an absorbing state). Note, however, that in principle u can be less than c – agents may receive a negative net flow payoff while in the working state, yet prefer to remain on the platform to obtain a positive net flow payoff while shirking.

In the next section, the principal will maximize her objective (three possibilities are explored) $V(u, w^0, w^*)$ over the set of feasible platforms. For tractability, most of the analysis that follows will focus on a particular specification of the function U (see Specification 1 below), but first, we briefly address an alternative specification which allows sufficient conditions for existence to be stated in closed form.[5]

Proposition 4.1 considers the case where μ_H and μ_L are the flow output values (not just signal drift values) of high and low effort, respectively, and each agent obtains the average (or expected, in the case of random matching) output as a flow payoff. Define

$$r^* := \frac{2(\mu_H - \mu_L)^2}{c}\left(c + \mu_H - 2\mu_L - 2\sqrt{(c-\mu_L)(\mu_H - \mu_L)}\right). \tag{4.4}$$

Proposition 4.1. *Let $U(Q) = Q\mu_H + (1-Q)\mu_L$ and for $\mu_L \leq c$. If $\mu_L > c$ or if $\mu_L \leq c$ and $r \in [0, r^*]$, then the feasible set is nonempty (i.e., there exists a triple (u, w^0, w^*) that satisfies (4.2) and (4.3)). The cutoff r^* is increasing in μ_H and decreasing in c.*

We now state a specific functional form for agents' flow payoffs from being on the platform, which allows us to more sharply describe the feasible set. In particular, the specification is that agents obtain a fixed positive flow benefit from being on the platform provided that a sufficiently high fraction of agents are working, and this flow benefit is exactly c. This ensures that agents' continuation values have upward drift since they only earn positive net flow payoffs while shirking.

Specification 1. *There exists a constant $\bar{Q} \in (0,1)$ such that each agent's flow utility from being on the platform while a fraction Q of the other agents are working is*

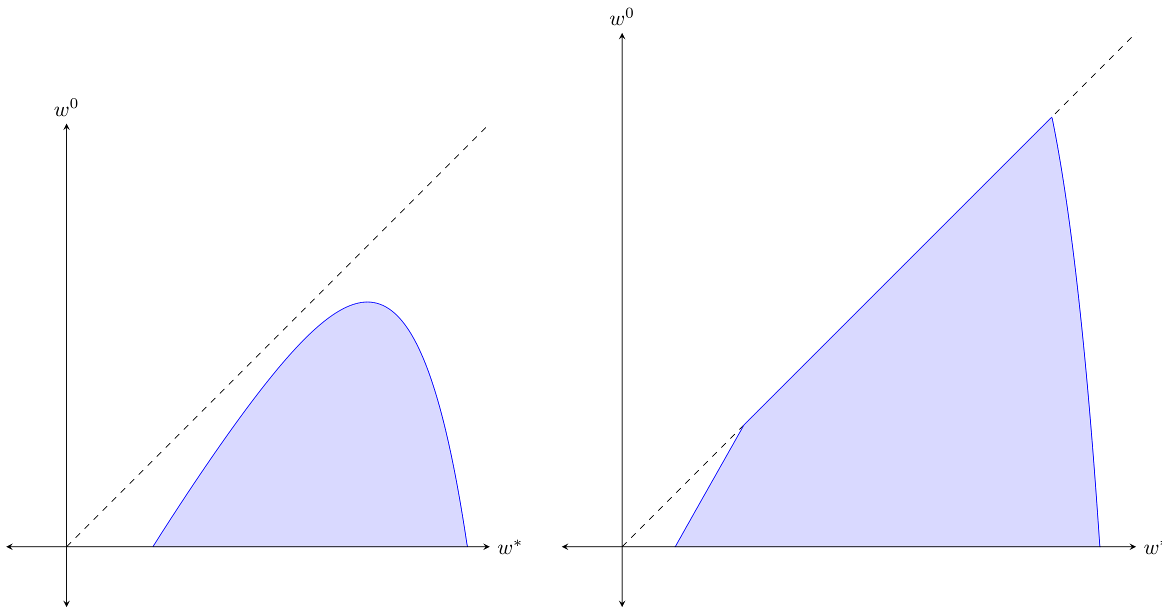
$$u(Q) = \begin{cases} c & \text{if } Q \geq \bar{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Under Specification 1 we can prove several claims about the shape of the feasible set (and, relevant to later results, about the relationship between the design parameters and the measures α and ν of working and shirking agents). First, we show that fixing w^*, the platform is feasible if and only if w^0 is sufficiently small; in other words, the feasible set lies under some curve $\bar{w}^0(w^*)$. Intuitively, if w^0 is lower, then fewer agents reach the shirking state w^* before being kicked off, and so the fraction of agents who work is higher. Moreover, this curve is differentiable where $\bar{w}^0(w^*) < w^*$. Next, we show that, fixing w^0, the fraction of working agents is a single-peaked function of w^*, and thus the horizontal cross sections of the feasible set are intervals. For small w^*, say very close to w^0, increasing w^* can improve the average effort on the platform by delaying the time at which new agents shirk. On the other hand, as w^* increases, agents must spend a longer amount of time shirking – the reflecting barrier at w^* becomes stickier. Trivially, the feasible set shrinks in the sense of set-inclusion

[5] Despite this advantage, the specification in Proposition 4.1 is much less tractable for our later purposes.

as $\bar{Q}$ increases; we also show that for sufficiently large $\bar{Q}$, the feasible set lies entirely below the 45-degree line. Finally, we note that a "wedge" always exists between the bottom left of the feasible set and the 45-degree line. Figure 2 shows the feasible set under Specification 1 for fixed values of the parameters (r, c, u, λ, ψ) and two values of $\bar{Q}$.

Proposition 4.2 (The Feasible Set)**.** *The feasible set can be written as* $\{(w^*, w^0) \in R^2_+ : w^0 \in (0, \bar{w}^0(w^*)]\}$, *where* $\bar{w}^0(w^*)$ *is a single-peaked function taking values in* $[0, w^*]$. *The feasible set is nonempty if* $\bar{Q}$ *is not too large. If* w^* *is such that* $\bar{w}^0(w^*) < w^*$, *then* $\bar{w}^0$ *is continuously differentiable at* w^*. *For sufficiently small* $w^0 > 0$, $w^* > w^0$ *for all feasible* (w^*, w^0). *For sufficiently large* $\bar{Q}$, *the feasible set lies strictly below the 45-degree line.*



(a) $\bar{Q} = .6$ (b) $\bar{Q} = .4$

Figure 2: Feasible set for $(r, c, u, \lambda, \psi) = (.05, .6, .6, 1.2, 1)$ and $\bar{Q} = .4, .6$.

To conclude this section, we show that a platform may not exist if agents are myopic.

Proposition 4.3 (Non-existence of a platform)**.** *Consider any non-decreasing* $U(Q)$. *If* $r = \infty$, *so agents are perfectly myopic, then a platform fails to exist.*

The intuition for this is straightforward. each agent is motivated to exert high effort by the threat of eventual termination and by the promise of eventual *vacation*. As he becomes perfectly impatient, the prospect of future sticks and carrots lose their salience, and it becomes impossible to incent high effort.

5 The Principal's Problem

We now solve the problem of the platform designer under three different objectives: maximizing per capita output, maximizing entry fees, and maximizing total platform size.

5.1 Maximizing Per Capita Output

We now consider a designer whose objective is to maximize steady-state output per capita. That is, the platform designer solves

$$\max_{(u,w^0,w^*)} Q(u,w^0,w^*)\mu_H + (1 - Q(u,w^0,w^*))\mu_L$$

subject to (4.2) and (4.3), which is equivalent to maximizing $Q(u,w^0,w^*)$. Since Q is decreasing in w^0, the optimal platform is degenerate – the platform designer would like to set w^0 "as close as possible" to 0 and the platform will have virtually no agents. If we extend the feasible set to allow $w^0 = 0$, then a solution exists, and the optimal w^* is interior, which is intuitive given the wedge result from Proposition 4.2 and the fact that that Q is a single-peaked function of w^*. Interestingly, Q remains less than 1: although new agents are arbitrarily unlikely to reach the shirking state, a positive fraction of agents remain shirking. Alternatively, this solution can be characterized as a supremum over feasible platforms with $w^0 > 0$. Define $V_{PC} = \sup_{(u,w^0,w^*)} Q(u,w^0,w^*)\mu_H + (1 - Q(u,w^0,w^*))\mu_L$ (PC for per-capita). Then there exists a unique pair $(w^0_{PC}, w^*_{PC}) = (0, w^*_{PC})$ such that for any sequence $\{(w^0_n, w^*_n)\}_{n=1}^{\infty}$ such that each (u, w^0_n, w^*_n) is feasible, $V(u, w^0_n, w^*_n) \to V_{PC}$ if and only if $(w^0_n, w^*_n) \to (0, w^*_{PC})$.

Proposition 5.1 (The optimal platform for per-capita surplus is degenerate)**.** *Under Specification 1, there exists a unique pair $(w^0_{PC}, w^*_{PC}) = (0, w^*_{PC})$ such that the platform designer maximizes per-capita surplus by setting (w^0, w^*) arbitrarily close to $(0, w^*_{PC})$.*

This result is reminiscent of partnership organizations in which senior (i.e., vested) partners recruit junior colleagues on the lowest rung of the ladder and promote virtually none of them. Interestingly we find that such an organization is itself *very* small in steady state, as partners trade off growth of the organization in order to maintain a high percentage of hard-working juniors.

Entry Fees

Suppose the platform designer can charge a fixed entry fee p from each new agent to the platform. Assuming agents have an outside option of 0, agents are willing to pay up to w^0 to join the platform. Recalling that ψ is the exogenous inflow rate of new agents, the platform designer can thus earn a maximum flow payoff of ψw^0. The design problem then reduces to maximizing $V(u, w^0, w^*) = w^0$ subject to (4.2) and (4.3).

By the single-peakedness result in Proposition 4.2, when the feasible set is nonempty there exists a unique feasible platform which solves the principal's problem; if there were two such feasible platforms, a third would exist that yields higher Q, and there would be room to increase w^0 while remaining in the feasible set. This platform is the top of the feasible set, as shown in Figure 2. If $\bar{Q}$ is sufficiently large that the feasible set lies strictly below the 45-degree line, then this solution must involve $w^0 < w^*$. That is, even though agents are likely to reach higher continuation values than w^0 while they are on the platform, the designer cannot start them at higher levels, because this would increase the fraction of agents who are shirking and violate feasibility.

Proposition 5.2. *Under Specification 1, if the feasible set is nonempty, there exists a unique platform (u, w^0_{EF}, w^*_{EF}), with $w^*_{EF} > w^*_{PC}$, that maximizes the platform designer's payoff from entry fees. For sufficiently large $\bar{Q}$, $w^0_{EF} < w^*_{EF}$.*

Platform Size

Suppose the platform designer wishes to maximize the total size of the platform, $V(u, w^0, w^*) = \alpha(u, w^0, w^*) + \nu(u, w^0, w^*)$. In the appendix (Lemmas B.1 and B.2), we show that both α and ν are increasing in both w^0 and w^* in the feasible set. Hence, any platform which maximizes the platform's total size must lie on the northeastern frontier of the feasible set: it must be of the form (u, w^0, w^*), where $w^* \geq w^*_{EF}$ and where $w^0 = \bar{w}^0(w^*)$. Moreover, if $\bar{Q}$ is sufficiently large, we argue that platform size must be maximized on the *interior* of this northeastern frontier. First, platform size cannot be maximized at the bottom right corner of the feasible set, since platform size vanishes as $w^0 \to 0$ (this argument does not depend on $\bar{Q}$ being sufficiently large). Second, if $\bar{Q}$ is sufficiently large, then platform size cannot be maximized at (u, w^0_{EF}, w^*_{EF}), because the designer can trade off a marginal reduction in w^0 for a (relatively) arbitrarily large increase in w^*. In other words, when $\bar{Q}$ is sufficiently large, the top of the feasible set is flat, and the indifference curve for platform size which intersects the top of the feasible set must cut into the feasible set, and there exist other points of the feasible set which yield a larger platform size.

Proposition 5.3. *Under Specification 1, if the feasible set is nonempty, platform size is maximized on the northeastern frontier of the feasible set. Moreover, if $\bar{Q}$ is sufficiently large, then platform size is maximized on the interior of this frontier.*

Numerically, it appears that there is a unique point in the feasible set which maximizes platform size.[6] Figure 2 shows a numerically solved example.

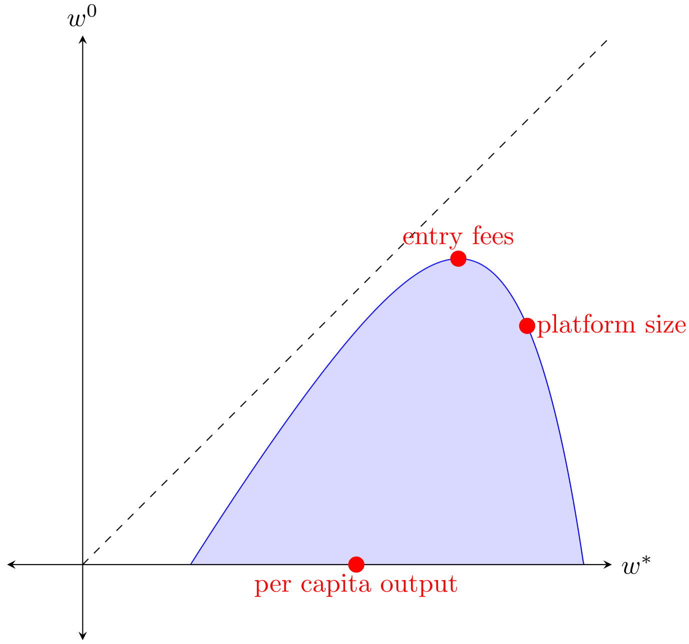


Figure 3: Numerical solutions to principal's problem under various objectives for $(r, c, u, \lambda, \psi, \bar{Q}) = (.05, .6, .6, 1.2, 1, .6)$. Here, the platforms that maximize per capita output, entry fees, and platform size have coordinates $(3.62, 0)$, $(4.97, 4.05)$ and $(5.88, 3.16)$, respectively.

6 Discussion

We have proposed a model of platform design in which there is a large number of small agents, whose efforts exert positive externalities but whose interactions are such that high effort can only be incentivized through a central reputation system. Absent transfer payments, agents must be permitted to shirk in some instances after good performance. Under the Brownian monitoring (or output) technology, this implies that agents' continuation values follow a sticky Brownian motion. Using techniques in stochastic calculus, we characterize the steady state of the platform as a stationary distribution over continuation values. Finally, we frame the principal's optimization problem in terms of the steady state distribution, where she optimizes platform design parameters subject to a feasibility constraint.

Our methods could be applied to settings with other kinds of dynamics. For example, although we have considered an effort-based model, it could be that the value to the platform

[6]This is the subject of ongoing analytical work.

of an agent is an exogenous type, about which the principal and each agent learn over time (through, say, Brownian diffusions with type-dependent drift). In this case, our steady state would be with respect to the reputation of each agent. At low reputations, the principal would remove agents from the platform, while at high reputations, agents might take an outside option.

Our model could also be extended to capture other aspects of real world platforms. For example, we have assumed an exogenous inflow rate of new agents, but one could endogenize the inflow rate, say to be an increasing function of the starting payoff w^0. We have also assumed that no agents voluntarily leave the platform, but there are several ways that agents could leave a platform in practice. Agents might have idiosyncratic shocks that force them to leave the platform, independent of their continuation values; we conjecture that this would simply increase the effective discount factor of agents, and would reduce the feasible set. A more substantively different possibility would be to give agents a positive outside option; this would put a positive lower bound on continuation values and would also restrict the feasible set.

Finally, we have assumed that agents are motivated by the threat of permanent expulsion. It, however, would be interesting to study a *closed platform* (i.e., with no exogenous inflows) in which poor performance resulted in temporary expulsion at some endogenously determined level of continuation utility $w_* > 0$ with reinsertion at w^0. We leave this and the other variations of the model discussed above for future work.

A Proofs for Section 3

Proof of Lemma 3.1. The infinitesimal generator of the W process is the operator Γ defined by

$$\Gamma h(w) = \lim_{dt \downarrow 0} \frac{\mathbb{E}_w[h(W_{dt})] - h(w)}{dt}.$$

This is a type of stochastic derivative, taken in expectation. For all $w > 0$ and functions h in a suitable domain, the above limit is well-defined and takes values:

$$\Gamma h(w) = \begin{cases} (rw^* - u)h'(w^*) & \text{if } w = w^* \\ (rw - (u-c))h'(w) + \frac{\lambda^2}{2}h''(w) & \text{if } w \in (0, w^*). \end{cases}$$

In particular, the above is valid for all h such that h is twice continuously differentiable and bounded and that $\Gamma h(w)$ is continuous, including at w^*. For $w = 0$, the generator is not defined since the jump from 0 to w^0 is instantaneous. For convenience, define $\mu(w) := rw - (u - c)$ and recall that $\rho(w) := rw - u$.

Now ν is a stationary distribution if and only if for all $t \geq 0$ and all functions h, we have

$$\int_{(0,w^*]} h(w)\nu(dw) = \int_{[0,w^*]} \mathbb{E}_w[h(W_t)]\nu(dw). \tag{A.1}$$

Essentially, this condition says that any statistic of a stationary distribution is unchanging over time. In order to characterize a steady-state distribution, we want to transform the right side of the above expression into terms involving only $h''(w)$ and $h'(w^*)$. Expanding the right hand side using Ito's formula, we have

$$\int_{(0,w^*]} \mathbb{E}_w[h(W_t)]\nu(dw) =$$
$$\int_{(0,w^*]} \left(h(w) + \mathbb{E}_w \left[\int_0^t \Gamma h(W_s)ds + \sum_{0<s\leq t} \Delta h(W)_s \right] \right) \nu(dw)$$

where $\Delta h(W)_s := h(W_s) - h(W_{s-})$ for $s > 0$. Subtracting the left hand side of (A.1) from this, we have

$$\begin{aligned} 0 &= \int_{(0,w^*]} \mathbb{E}_w \left[\int_0^t \Gamma h(W_s)\, ds + \sum_{0<s\leq t} \Delta h(W)_s \right] \nu(dw) \\ &= \int_{(0,w^*]} \mathbb{E}_w \left[\int_0^t \Gamma h(W_s) \right] ds\nu(dw) + \int_{(0,w^*]} \mathbb{E}_w \left[\sum_{0<s\leq t} \Delta h(W)_s \right] \nu(dw) \end{aligned}$$

Dividing through by t and taking limits as $t \to 0$ yields

$$\begin{aligned} 0 &= \int_{(0,w^*]} \Gamma h(w)\nu(dw) + \lim_{t\to 0} \frac{1}{t} \int_{[0,w^*]} \mathbb{E}_w \left[\sum_{0<s\leq t} \Delta h(W)_s \right] \nu(dw) \\ &= \int_{(0,w^*]} \Gamma h(w)\nu(dw) + \frac{\lambda^2}{2} f'_-(0)(h(w^0) - h(0)). \end{aligned} \tag{A.2}$$

To obtain the second term of (A.2), $\Delta h(W)_s > 0$ only when $W_{s-} = \lim_{t\to s} W_t = 0$, and in these cases we have $\Delta h(W)_s = h(W_s) - h(W_{s-}) = h(w^0) - h(0)$. To a first order approximation, the second term of (A.2) is thus the expectation, over starting points w, of the size of a single jump, $h(w^0) - h(0)$, times the probability that the process starting from w reaches 0 (the probability of 2 or more jumps may be ignored since once the process resets at w^0 it is very far away from 0). Thus the second term above is $(h(w^0) - h(0))\frac{\lambda^2 f'_-(0)}{2}$.

The first term of (A.2) can be expanded as

$$\int_{(0,w^*)}\left[\mu(w)h'(w)+\frac{\lambda^2}{2}h''(w)\right]\nu(dw)+\rho(w^*)h'(w^*)\nu\{w^*\}$$
$$=\int_{(0,w^*)}\mu(w)h'(w)\nu(dw)+\int_{(0,w^*)}\frac{\lambda^2}{2}h''(w)\nu(dw)+\rho(w^*)h'(w^*)\nu\{w^*\} \tag{A.3}$$

We now focus on the first term of (A.3). The integral can be split into two regions, $(0,w^0)$ and $[w^0,w^*)$. Then, by writing $h'(w)=h'(w^0)-\int_w^{w^0}h''(y)dy$, the first term of (A.3) is equivalent to

$$\begin{aligned}
&\int_{(0,w^0)}\mu(w)\left[h'(w^0)-\int_w^{w^0}h''(y)dy\right]f_-(w)dw\\
&\qquad+\int_{[w^0,w^*)}\mu(w)\left[h'(w^0)+\int_{w^0}^{w}h''(y)dy\right]f_+(w)dw\\
&=h'(w^0)\int_{(0,w^*)}\mu(w)\nu(dw)-\int_0^{w^0}\left[\int_0^w\mu(y)f_-(y)dy\right]h''(w)dw\\
&\qquad+\int_{[w^0,w^*)}\left[\int_{[w,w^*)}\mu(y)f_+(y)dy\right]h''(w)dw
\end{aligned}$$

Collecting terms thus far, the first term of (A.2) becomes

$$\begin{aligned}
\int_{(0,w^*]}\Gamma h(w)\nu(dw)=&\int_{(0,w^0)}h''(w)\left[\frac{\lambda^2}{2}f_-(w)-\int_0^w\mu(y)f_-(y)dy\right]dw\\
&+\int_{[w^0,w^*)}h''(w)\left[\frac{\lambda^2}{2}f_+(w)+\int_{[w,w^*)}\mu(y)f_+(y)dy\right]dw\\
&+h'(w^0)\int_{(0,w^*)}\mu(w)\nu(dw)\\
&+\rho(w^*)h'(w^*)\nu\{w^*\}.
\end{aligned} \tag{A.4}$$

The first two terms of (A.4), having integrals involving $h''(w)$ as coefficients, are all set. Take the last two terms of (A.4) and add back in the second term on the RHS of (A.2) to obtain the expression

$$\frac{\lambda^2}{2}f'_-(0)(h(w^0)-h(0))+h'(w^0)\int_{(0,w^*)}\mu(w)\nu(dw)+\rho(w^*)h'(w^*)\nu\{w^*\}. \tag{A.5}$$

As noted, the goal is to transform as much of the h involvement above into h'' and $h'(w^*)$ terms. For the first term of (A.5), integrate the derivatives twice and exchange the order of integration to get

$$\begin{aligned}
\frac{\lambda^2}{2}f'_-(0)(h(w^0)-h(0)) &= \frac{\lambda^2}{2}f'_-(0)\int_0^{w^0} h'(w)dw \\
&= \frac{\lambda^2}{2}f'_-(0)\int_0^{w^0}\left[h'(w^*)-\int_w^{w^*} h''(y)dy\right]dw \\
&= \frac{\lambda^2}{2}f'_-(0)\left(w^0h'(w^*)-\int_0^{w^0}\left[\int_w^{w^*} h''(y)dy\right]dw\right) \\
&= \frac{\lambda^2}{2}f'_-(0)\left(w^0h'(w^*)-\int_0^{w^0} wh''(w)dw-\int_{w^0}^{w^*} w_0h''(w)dw\right)
\end{aligned} \tag{A.6}$$

For the second term of (A.5), we have

$$\begin{aligned}
h'(w^0)\int_{(0,w^*)}\mu(w)\nu(dw) &= \left(h'(w^*)-\int_{w^0}^{w^*} h''(y)dy\right)\int_{(0,w^*)}\mu(w)\nu(dw) \\
&= h'(w^*)\int_{(0,w^*)}\mu(w)\nu(dw)-\int_{w^0}^{w^*} h''(w)\left[\int_{(0,w^*)}\mu(y)\nu(dy)\right]dw
\end{aligned} \tag{A.7}$$

where we swap w and y as variables of integration for later convenience. Plugging (A.6) and (A.7) back into (A.5) and adding back in the first two terms of (A.4), we can write (A.2) as

$$0 = h'(w^*)M^* + \int_{(0,w^0)} h''(w)M_-(w)dw + \int_{[w^0,w^*)} h''(w)M_+(w)dw, \tag{A.8}$$

where we define

$$\begin{aligned}
M^* &:= \frac{\lambda^2}{2}f'_-(0)w^0 + \int_{(0,w^*)}\mu(w)\nu(dw)+\rho(w^*)\nu\{w^*\} \\
M_-(w) &:= -\frac{\lambda^2}{2}f'_-(0)w+\frac{\lambda^2}{2}f_-(w)-\int_0^w \mu(y)f_-(y)dy \\
M_+(w) &:= -\frac{\lambda^2}{2}f'_-(0)w^0+\frac{\lambda^2}{2}f_+(w)+\int_w^{w^*}\mu(y)f_+(y)dy-\int_{(0,w^*)}\mu(y)\nu(dy).
\end{aligned}$$

Equation (A.8) is exactly what we are after. It allows us to completely characterize the steady-state distribution. Specifically, because $h'(w^*)$ and $h''(w)$ are completely free (up to the differentiability conditions), the expressions attached to them must all vanish:

$$M^* = 0$$
$$M_-(w) \equiv 0$$

$$M_+(w) \equiv 0.$$

Equation (A.8) thus implies the following:

1. From $M''_-(w) = 0$ and $M''_+(w) = 0$, we recover the Kolmogorov forward equation (3.1) on the left and right pieces.

2. $M_-(0+) = 0$ gives $f_-(0+) = 0$.

3. $M_-(w^0) = M_+(w^0)$ implies $f_-(w^0) = f_+(w^0)$.

4. $M'_-(w^0) = M'_+(w^0)$ implies $f'_-(0) = f'_-(w^0) - f'_+(w^0)$.

5. $0 = M^* + M_+(w^*)$ implies $\frac{\lambda^2}{2} f_+(w^*-) + \rho(w^*)\nu\{w^*\} = 0$.

6. $\frac{\lambda^2}{2} f'_-(0) = \psi$.

□

Proof of Proposition 3.1. As shown above, the steady state distribution of agents can be described by the equations

$$f_\pm(w) = e^{\gamma(w)^2}(C_1^\pm + C_2^\pm \text{erf}\,\{\gamma(w)\})$$

subject to the constraints derived immediately above, where $\gamma(w) := \frac{rw-(u-c)}{\lambda\sqrt{r}}$ and $\text{erf}\,\{z\} := \frac{2}{\sqrt{u}} \int_0^z e^{-t^2} dt$ is the Gauss error function.

As $f_-(0) = 0$, we have:

$$0 = e^{\frac{(u-c)^2}{\lambda^2 r}} \left(C_1^- + C_2^- \text{erf}\left\{ \frac{-(u-c)}{\lambda\sqrt{r}} \right\} \right)$$

Because the error function has odd symmetry, this means that

$$C_1^- - C_2^- \text{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} = 0$$

Knowing too that $f'_-(0) = \frac{2\psi}{\lambda^2}$, and by differentiating $f_-(w)$, we get:

$$\begin{aligned}
\frac{2\psi}{\lambda^2} &= 2\gamma(0)\gamma'(0)e^{\gamma(0)^2} \left(C_2^- \text{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} + C_2^- \text{erf}\,\{\gamma(0)\} \right) + e^{\gamma(0)^2} C_2^- \gamma'(0)\text{erf}'\{\gamma(0)\} \\
&= 2e^{\gamma(0)^2} C_w^- \gamma'(0) \frac{e^{-\gamma(0)^2}}{\sqrt{u}} = \frac{2C_2^- \sqrt{r}}{\lambda\sqrt{u}} \\
&\Rightarrow C_2^- = \frac{\psi\sqrt{u}}{\lambda\sqrt{r}}
\end{aligned}$$

Thus the lower segment of the distribution function is

$$f_-(w) = e^{\gamma(w)^2} \left(\frac{\psi\sqrt{u}}{\lambda\sqrt{r}} \left[\text{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} + \text{erf}\left\{\gamma(w)\right\} \right] \right) \tag{A.9}$$

Since $f_-(w)$ and $f_+(w)$ must agree at w^0, we set the lower and upper f functions equal at w^0 to get

$$\begin{aligned} f_+(w^0) = f_-(w^0) &= e^{\gamma(w^0)^2} \left(\frac{\psi\sqrt{u}}{\lambda\sqrt{r}} \left[\text{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} + \text{erf}\left\{\gamma(w^0)\right\} \right] \right) \\ &= e^{\gamma(w^0)^2}(C_1^+ + C_2^+ \text{erf}\left\{\gamma(w^0)\right\}) \end{aligned}$$

and thus, by rearranging terms, we find that

$$C_1^+ = \frac{\psi\sqrt{u}}{\lambda\sqrt{r}} \left[\text{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} + \text{erf}\left\{\gamma(w^0)\right\} \right] - C_2^+ \text{erf}\left\{\gamma(w^0)\right\}$$

Therefore,

$$\begin{aligned} &f_+(w) = \\ &\quad e^{\gamma(w)^2} \left(\frac{\psi\sqrt{u}}{\lambda\sqrt{r}} \left[\text{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} + \text{erf}\left\{\gamma(w^0)\right\} \right] - C_2^+ \text{erf}\left\{\gamma(w^0)\right\} + C_2^+ \text{erf}\left\{\gamma(w)\right\} \right) \end{aligned}$$

and, differentiating both $f_+(w)$ and $f_-(w)$, we get

$$f_+'(w) = 2\gamma(w)\gamma'(w)f_+(w) + e^{\gamma(w)^2} C_2^+ \text{erf}'\{\gamma(w)\}\gamma'(w)$$

and

$$f_-'(w) = 2\gamma(w)\gamma'(w)f_-(w) + \frac{\psi\sqrt{u}}{\lambda\sqrt{r}} e^{\gamma(w)^2} \text{erf}'\{\gamma(w)\}\gamma'(w)$$

Because $f_-'(0) = f_-'(w^0) - f_+'(w^0)$, it must be that

$$C_2^+ = \frac{\frac{\psi\sqrt{u}}{\lambda\sqrt{r}} \left[e^{\gamma(w^0)} \text{erf}'\{\gamma(w^0)\}\gamma'(w^0) - e^{\gamma(0)^2} \text{erf}'\{\gamma(0)\}\gamma'(0) \right]}{e^{\gamma(w^0)^2} \text{erf}'\{\gamma(w^0)\}\gamma'(w^0)} = 0$$

Thus, the upper segment of the distribution function is

$$f_+(w) = e^{\gamma(w)^2} \frac{\psi\sqrt{u}}{\lambda\sqrt{r}} \left[\text{erf}\left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} + \text{erf}\left\{\gamma(w^0)\right\} \right] \tag{A.10}$$

Finally, completing the derivation of the distribution of agents in the steady state, the mass of agents at w^*, $\nu\{w^*\}$, satisfies

$$\nu\{w^*\} = \frac{\lambda^2 f_+(w^*)}{2(u - rw^*)}.$$

□

B Proofs for Section 4

Proof of Proposition 4.1. Let $Q(u, w^0, w^*) := \frac{\alpha(u,w^0,w^*)}{\alpha(u,w^0,w^*)+\nu(u,w^0,w^*)}$, the fraction of agents working. Taking $w^0 \to 0$ and using L'Hôpital's rule,

$$\begin{aligned}
\lim_{w^0\to 0} Q(u, w^0, w^*) &= \frac{\frac{\partial}{\partial w^0}\alpha(u, w^0, w^*)}{\frac{\partial}{\partial w^0}\alpha(u, w^0, w^*) + \frac{\partial}{\partial w^0}\nu(u, w^0, w^*)}\Big|_{w^0=0} \\
&= \frac{\int_0^{w^*} \exp(\gamma(w)^2)\, dw}{\int_0^{w^*} \exp(\gamma(w)^2)\, dw + \frac{\lambda^2}{2(u-rw^*)} \exp(\gamma(w^*)^2)} \\
&=: \hat{Q}(u, w^*).
\end{aligned}$$

Taking $u \downarrow rw^*$, we have

$$\begin{aligned}
\lim_{u\downarrow rw^*} \hat{Q}(u, w^*) &= \frac{\int_0^{w^*} \exp\left(\frac{(r(w-w^*)+c)^2}{\lambda^2 r}\right) dw}{\int_0^{w^*} \exp\left(\frac{(r(w-w^*)+c)^2}{\lambda^2 r}\right) dw + \infty \cdot \exp\left(\frac{c^2}{\lambda^2 r}\right)} \\
&= 0.
\end{aligned}$$

Taking $u \to rw^* + c < \mu_H$, so that $\gamma(w^*) \to 0$, we have

$$\lim_{u\to rw^*+c} \hat{Q}(u, w^*) = \frac{\int_0^{w^*} \exp(r(w - w^*)^2/\lambda^2)\, dw}{\int_0^{w^*} \exp(r(w - w^*)^2/\lambda^2)\, dw + \frac{\lambda^2}{2c}}.$$

Abusing notation, let $\hat{Q}(w^*)$ denote the fraction above. For a fixed point, it suffices to find $w^* \in (0, (\mu_H - c)/r)$ such that

$$\hat{Q}(w^*)\mu_H + (1 - \hat{Q}(w^*))\mu_L > rw^* + c,$$

which is equivalent to

$$\int_0^{w^*} \exp(r(w - w^*)^2/\lambda^2)\, dw(\mu_H - c - rw^*) + \frac{\lambda^2}{2c}(\mu_L - c - rw^*) > 0.$$

The integrand above is bounded below by 1, so the whole expression on the left side is bounded below by

$$w^*(\mu_H - c - rw^*) + \frac{\lambda^2}{2c}(\mu_L - c - rw^*).$$

Letting x stand in for w^*, this expression is a concave quadratic function of x,

$$\begin{aligned} g(x) &:= Ax^2 + Bx + C, \text{ where} \\ A &:= -r \\ B &:= \left(\mu_H - c - \frac{\lambda^2 r}{2c}\right) \\ C &:= \frac{\lambda^2}{2c}(\mu_L - c). \end{aligned}$$

Thus a sufficient condition for a fixed point is that there exists $x > 0$ such that $g(x) > 0$. Note that if $\mu_L > c$, then $g(0) > 0$ and we are done. For the rest of the proof, assume $\mu_L < c$, so that $g(0) < 0$. It follows that if g has real roots, either both are positive or both are negative; if both roots are positive, we are done. Since both roots have the same sign, both are positive if and only if their sum is positive. Now g has real roots, the sum of which is positive, if and only if both of the following conditions hold:

$$\begin{aligned} & 0 < B^2 - 4AC = \left(\mu_H - c - \frac{\lambda^2 r}{2c}\right)^2 + 4r\frac{\lambda^2}{2c}(\mu_L - c) && \text{(B.1)} \\ & 0 < -\frac{B}{A} \\ \iff & 0 < B = \mu_H - c - \frac{\lambda^2 r}{2c} && \text{(B.2)} \end{aligned}$$

Using $\lambda = \frac{c}{\mu_H - \mu_L}$, these inequalities expand to

$$0 < \left(\mu_H - c - \frac{cr}{2(\mu_H - \mu_L)^2}\right)^2 + 2\frac{cr}{(\mu_H - \mu_L)^2}(\mu_L - c) \quad \text{(B.3)}$$

$$0 < \mu_H - c - \frac{cr}{2(\mu_H - \mu_L)^2} \quad \text{(B.4)}$$

Collecting r terms, the right side of (B.3) is a convex quadratic in r,

$$h(r) := \frac{c^2}{4(\mu_H - \mu_L)^4}r^2 + \left(\frac{-c^2 + 2c\mu_L - c\mu_H}{(\mu_H - \mu_L)^2}\right)r + (\mu_H - c)^2$$

with sign pattern $+,-,+$. It follows that $h(0)>0$ and $h'(0)<0$. The inequality (B.4) is equivalent to

$$r<\bar{r}:=\frac{2(\mu_H-c)(\mu_H-\mu_L)^2}{c}.$$

For $r=\bar{r}$, the first term on the right side of (B.3) vanishes while the second term is negative, so $h(\bar{r})<0$. It follows that h has two real roots, both positive. Now $h(r)$ is decreasing for all $r\in[0,\bar{r}]$ and in this interval, $h(r)\geq 0$ if and only if $r<r^*$, where r^* is the lower of the two roots of h, given explicitly by the formula (4.4).

We claim that r^* is increasing in μ_H and decreasing in c. For μ_H it suffices to show that the term $c+\mu_H-2\mu_L-2\sqrt{(c-\mu_H)(\mu_H-\mu_L)}$ is increasing in μ_H. By direct computation, its derivative w.r.t. μ_H is $1-\frac{\sqrt{c-\mu_L}}{\sqrt{\mu_H-\mu_L}}>0$ as $\mu_H>c$. For c, we have

$$\begin{aligned}
&\frac{\partial}{\partial c}r^*= \\
&\quad\frac{2(\mu_H-\mu_L)^2}{c^2}\left(-\mu_H+2\sqrt{(c-\mu_L)(\mu_H-\mu_L)}+2\mu_L-c\frac{\mu_H-\mu_L}{\sqrt{(c-\mu_L)(\mu_H-\mu_L)}}\right)\\
&\quad<\frac{2(\mu_H-\mu_L)^2}{c^2}\left(-\mu_H+2\sqrt{(c-\mu_L)(\mu_H-\mu_L)}+2\mu_L-c\right)\\
&\quad=\frac{4(\mu_H-\mu_L)^2}{c^2}\left(\sqrt{(c-\mu_L)(\mu_H-\mu_L)}-\frac{(\mu_H-\mu_L)+(c-\mu_L)}{2}\right)
\end{aligned}$$

which is negative by applying the Arithmetic Mean–Geometric Mean inequality to the pair of positive numbers $(c-\mu_L,\mu_H-\mu_L)$. $\square$

Lemma B.1. *Under Specification 1, both $\alpha(u,w^0,w^*)$ and $\nu(u,w^0,w^*)$ are increasing in w^0 and $Q(u,w^0,w^*)$ is strictly decreasing in w^0 for feasible (u,w^0,w^*).*

Proof. We have $Q=\frac{\alpha}{\alpha+\nu}=\frac{1}{1+\frac{\nu}{\alpha}}$ which is decreasing iff $\frac{\nu}{\alpha}$ is increasing, which is true iff $\frac{\nu_{w^0}}{\nu}>\frac{\alpha_{w^0}}{\alpha}$. Using $X:=\frac{u-c}{\lambda\sqrt{r}}$ [7] and $Y:=\frac{\psi\sqrt{u}}{\lambda\sqrt{r}}$ and expanding, these quantities are

$$\begin{aligned}
\nu&=\frac{\lambda^2}{2(u-rw^*)}f_+(w^*)=\frac{\lambda^2}{2(u-rw^*)}e^{\gamma(w^*)^2}Y\left(\operatorname{erf}\{X\}+\operatorname{erf}\{\gamma(w_0)\}\right)\\
\nu_{w^0}&=\frac{\psi}{u-rw^*}e^{\gamma(w^*)^2-\gamma(w^0)^2}\\
\alpha&=\int_0^{w^0}f_-(w)\,dw+\int_{w^0}^{w^*}f_+(w)\,dw\\
&=\int_0^{w^0}e^{\gamma(w)^2}Y\left[\operatorname{erf}\{X\}+\operatorname{erf}\{\gamma(w)\}\right]dw+\int_{w^0}^{w^*}e^{\gamma(w)^2}Y\left[\operatorname{erf}\{X\}+\operatorname{erf}\left\{\gamma(w^0)\right\}\right]dw
\end{aligned}$$

[7] Under Specification 1, $X=0$ and thus $\operatorname{erf}\{X\}=0$, but we keep u and c distinct to show how they influence the various terms.

$$
\begin{aligned}
\alpha_{w^0} &= \int_{w^0}^{w^*} e^{\gamma(w)^2} Y \frac{2e^{-\gamma(w^0)^2}}{\sqrt{u}} \frac{\sqrt{r}}{\lambda} \, dw \\
&= \frac{2\sqrt{r} e^{-\gamma(w^0)^2}}{\lambda \sqrt{u} \left[\operatorname{erf}\{X\} + \operatorname{erf}\{\gamma(w^0)\}\right]} \int_{w^0}^{w^*} f_+(w) \, dw
\end{aligned}
$$

Define $Z := \frac{2\sqrt{r}e^{-\gamma(w^0)^2}}{\lambda\sqrt{u}[\operatorname{erf}\{X\}+\operatorname{erf}\{\gamma(w^0)\}]}$ to be the constant outside the integral. By canceling terms, we have $\frac{\nu_{w^0}}{\nu} = Z$, whereas

$$
\frac{\alpha_{w^0}}{\alpha} = Z \frac{\int_{w^0}^{w^*} f_+(w) \, dw}{\int_0^{w^0} f_-(w) \, dw + \int_{w^0}^{w^*} f_+(w) \, dw} < Z, \tag{B.5}
$$

so we are done. $\square$

Lemma B.2. *Under Specification 1, both $\alpha(u, w^0, w^*)$ and $\nu(u, w^0, w^*)$ are strictly increasing in w^* for feasible (u, w^0, w^*).*

Proof. Immediately, we have $\alpha_{w^*}(u, w^0, w^*) = f_+(w^*) > 0$. Next,

$$
\begin{aligned}
\nu_{w^*}(u, w^0, w^*) &= \frac{\partial}{\partial w^*} \left[\frac{\lambda^2}{2} \frac{f_+(w^*)}{u - rw^*} \right] \\
&= \frac{\lambda^2}{2} \frac{(u - rw^*) f'_+(w^*) + rw^* f_+(w^*)}{(u - rw^*)^2}.
\end{aligned}
$$

Now $f'_+(w^*) = \frac{\partial}{\partial w^*} e^{\gamma(w^*)^2} Y(\operatorname{erf}\{X\} + \operatorname{erf}\{\gamma(w^0)\}) > 0$, where X and Y are defined in the proof of Lemma B.1, and the rest of the numerator above is positive, so we conclude that $\nu_{w^*}(u, w^0, w^*) > 0$. $\square$

The next lemma implies that for any fixed $w^0 > 0$, there is a single (possibly empty) interval of w^* values for which the platform is feasible.

Lemma B.3. *Under Specification 1, for all fixed $w^0 > 0$, Q is single-peaked in w^*; that is, if $Q_{w^*} = 0$ for some particular value of w^*, then Q_{w^*} is decreasing at that point.*

Proof. To abbreviate, we use prime notation to denote the derivatives with respect to w^*. Now $Q = \frac{\alpha}{\alpha+\nu} = g\left(\frac{\nu}{\alpha}\right)$ where $g(x) := \frac{1}{1+x}$. Taking derivatives, we have $Q' = g'\left(\frac{\nu}{\alpha}\right)\left(\frac{\nu}{\alpha}\right)'$, which is zero if and only if $\left(\frac{\nu}{\alpha}\right)' = 0$. Moreover, $Q'' = g''\left(\frac{\nu}{\alpha}\right)\left[\left(\frac{\nu}{\alpha}\right)'\right]^2 + g'\left(\frac{\nu}{\alpha}\right)\left(\frac{\nu}{\alpha}\right)''$. Recall from Lemma B.2 that α' and ν' are positive. Using $Q' = 0$, we have $Q'' < 0$ if and only if

$$\begin{aligned} &\left(\frac{\nu}{\alpha}\right)'' > 0 \\ \iff\ & \alpha^2\nu'' - \alpha\nu\alpha'' > 2\alpha\alpha'\nu' - 2\nu(\alpha')^2 \\ \iff\ & \frac{\nu''}{\nu'} > \frac{\alpha''}{\alpha'}, \end{aligned}$$

where we have used $\alpha = \alpha' \frac{\nu}{\nu'}$. Next, recall that

$$\begin{aligned} \nu &= \frac{\lambda^2}{2}\frac{f_+}{u - rw^*} \\ \implies \nu' &= \frac{\lambda^2}{2}\left[\frac{(u - rw^*)f_+' + rf_+}{(u - rw^*)^2}\right] \\ \implies \nu'' &= \frac{\lambda^2}{2}\left[\frac{(u - rw^*)^3 f_+'' + 2rf_+'(u - rw^*)^2 + 2r^2 f_+(u - rw^*)}{(u - rw^*)^4}\right] \\ &> \frac{\lambda^2}{2}\left[\frac{(u - rw^*)f_+'' + rf_+'}{(u - rw^*)^2}\right]. \end{aligned}$$

Also recall that $\alpha' = f_+$ and $\alpha'' = f_+'$. Thus a sufficient condition for $\frac{\nu''}{\nu'} > \frac{\alpha''}{\alpha'}$ is

$$\begin{aligned} \frac{(u - rw^*)f_+'' + rf_+'}{f_+'(u - rw) + rf_+} &> \frac{f_+'}{f_+} \\ \iff \frac{f_+''}{f_+'} &> \frac{f_+'}{f_+}. \end{aligned}$$

Recalling that $f_+(w^*) = e^{\gamma(w^*)^2} Y \mathrm{erf}\left\{\gamma(w^0)\right\}$, this inequality simplifies to

$$\frac{1 + 2\frac{r(w^*)^2}{\lambda^2}}{w^*} > 2\frac{rw^*}{\lambda^2},$$

which clearly holds. We conclude that $Q' = 0$ implies $Q'' < 0$, so Q is single-peaked in w^*. □

We also prove the following lemma that shows the existence of a "wedge" in the graph of the feasible set.

Lemma B.4. *For sufficiently small w^0, any feasible platform requires $w^* > w^0$.*

Proof. We show that $\lim_{w^0 \to 0} Q(u, w^0, w^0) = 0$. We have $Q(u, w^0, w^0) = \frac{\alpha(u,w^0,w^0)}{\alpha(u,w^0,w^0)+\nu(u,w^0,w^0)}$

and we show that $\lim_{w^0\to 0}\frac{\alpha(u,w^0,w^0)}{\nu(u,w^0,w^0)}\to 0$. Expanding,

$$\lim_{w^0\to 0}\frac{\alpha(u,w^0,w^0)}{\nu(u,w^0,w^0)}=\lim_{w^0\to 0}\frac{\int_0^{w^0}f_-(w)\,dw}{\frac{\lambda^2}{2(u-rw^0)}f_-(w^0)}$$
$$=\frac{\lim_{w^0\to 0}f_-(w^0)}{\lim_{w^0\to 0}\frac{d}{dw}\left[\frac{\lambda^2}{2(u-rw^0)}f_-(w^0)\right]}.$$

The numerator has limit $f_-(0)=0$, while the denominator has limit

$$\lim_{w^0\to 0}Ye^{\gamma(w^0)^2}\left[2\gamma(w^0)\gamma'(w^0)\left(\mathrm{erf}\left\{X\right\}+\mathrm{erf}\left\{\gamma(w^0)\right\}\right)+\frac{2}{\sqrt{u}}e^{-\gamma(w^0)^2}\gamma'(w^0)\right]>0,$$

and it follows that $\lim_{w^0\to 0}\frac{\alpha(u,w^0,w^0)}{\nu(u,w^0,w^0)}=0$, as desired. □

Proof of Proposition 4.3. We know $u>rw^*$, and u is bounded above. Thus as $r\to\infty$, it must be that $w^*\to 0$ sufficiently quickly for rw^* to remain finite in the limit. However, as $w^*=0$ is infeasible, no platform exists for perfectly myopic agents. □

Lemma B.5. *For sufficiently small w^0, $Q_{w^*}(u,w^0,w^*)|_{w^*=w^0}>0$.*

Proof. We have $Q_{w^*}>0$ if and only if $\frac{\nu(u,w^0,w^*)}{\alpha(u,w^0,w^*)}$ is decreasing in w^*, or equivalently

$$\frac{\nu_{w^*}(u,w^0,w^*)}{\nu(u,w^0,w^*)}<\frac{\alpha_{w^*}(u,w^0,w^*)}{\alpha(u,w^0,w^*)}$$
$$\iff\quad\frac{\frac{(u-rw^*)f_+'(w^*)+rf_+(w^*)}{(u-rw^*)^2}}{\frac{f_+(w^*)}{(u-rw^*)}}<\frac{f_+(w^*)}{\int_0^{w^0}f_-(w)dw+\int_{w^0}^{w^*}f_+(w)dw}.$$

After simplifying, the above holds at $w^*=w^0$ if and only if

$$\frac{(u-rw^0)f_-'(w^0)+rf_-(w^0)}{f_-(w^0)(u-rw^0)}<\frac{f_-(w^0)}{\int_0^{w^0}f_-(w)dw}$$
$$\iff\quad\frac{r}{u}<\frac{f_-(w^0)^2-f_-'(w^0)\int_0^{w^0}f_-(w)dw}{f_-(w^0)\int_0^{w^0}f_-(w)dw}.\tag{B.6}$$

Now use L'Hopital's rule twice to evaluate the limit of the RHS of (B.6) as $w^0\downarrow 0$:

$$\lim_{w^0\downarrow 0}\frac{f_-(w^0)^2-f_-'(w^0)\int_0^{w^0}f_-(w)dw}{f_-(w^0)\int_0^{w^0}f_-(w)dw}=\lim_{w^0\downarrow 0}\frac{f_-(w^0)f_-'(w^0)-f_-''(w^0)\int_0^{w^0}f_-(w)dw}{f_-'(w^0)\int_0^{w^0}f_-(w)dw+f_-(w^0)^2}$$
$$=\lim_{w^0\downarrow 0}\frac{f_-'(w^0)^2-f_-'''(w^0)\int_0^{w^0}f_-(w)dw}{f_-''(w^0)\int_0^{w^0}f_-(w)dw+3f_-(w^0)f_-'(w^0)}$$

$$= +\infty.$$

It follows that (B.6) holds for sufficiently small w^0, and we conclude that $Q_w^*(u, w^0, w^*)|_{w^*=w^0} > 0$ for sufficiently small w^0. $\square$

Define $j(w^*) = \lim_{w^0 \to 0} Q(u, w^0, w^*)$; that this limit is well-defined is shown in the proof of the following lemma.

Lemma B.6. *The limit* $\lim_{w^0 \to 0} Q(u, w^0, w^*)$ *is well-defined, and* $j(w^*)$ *is a differentiable, single-peaked function which is maximized at some* $w^*_{PC} \in (0, u/r)$. *Moreover,* $j(w^*_{PC}) < 1$.

Proof. First, compute the limit as $w^0 \downarrow 0$ of $Q(u, w^0, w^*)$ directly as

$$\begin{aligned}
&\lim_{w^0 \downarrow 0} \frac{\alpha(u, w^0, w^*)}{\alpha(u, w^0, w^*) + \nu(u, w^0, w^*)} \\
&= \lim_{w^0 \downarrow 0} \frac{\int_0^{w^0} e^{\gamma(w)^2} Y \mathrm{erf}\left\{\gamma(w)\right\} dw + \int_{w^0}^{w^*} e^{\gamma(w)^2} Y \mathrm{erf}\left\{\gamma(w_0)\right\} dw}{\int_0^{w^0} e^{\gamma(w)^2} Y \mathrm{erf}\left\{\gamma(w)\right\} dw + \int_{w^0}^{w^*} e^{\gamma(w)^2} Y \mathrm{erf}\left\{\gamma(w_0)\right\} dw + \frac{\lambda^2}{2(u - rw^*)} e^{\gamma(w^*)^2} Y \mathrm{erf}\left\{\gamma(w_0)\right\}} \\
&= \frac{\int_0^{w^*} e^{\gamma(w)^2} dw}{\int_0^{w^*} e^{\gamma(w)^2} dw + \frac{\lambda^2}{2(u - rw^*)} e^{\gamma(w^*)^2}} \\
&=: j(w^*).
\end{aligned}$$

It is clear that w^* is differentiable. We argue that $j(w^*)$ is single-peaked, i.e., that $j''(w^*) < 0$ whenever $j'(w^*) = 0$. Define $\alpha_0(w^*) := \int_0^{w^*} e^{\gamma(w)^2} dw$ and $\nu_0(w^*) := \frac{\lambda^2}{2(u - rw^*)} e^{\gamma(w^*)^2}$, so that $j(w^*) = \frac{\alpha_0(w^*)}{\alpha_0(w^*) + \nu_0(w^*)}$. By arguments in the proof of Lemma B.3, it is enough to show that $\frac{\nu_0''}{\nu_0'} > \frac{\alpha_0''}{\alpha_0'}$ whenever $\left(\frac{\nu_0}{\alpha_0}\right)' = 0$, i.e., whenever $\alpha_0 \nu_0' = \nu_0 \alpha_0'$. Define $f_0(w^*) := e^{\gamma(w^*)^2}$. The rest of the proof of single-peakedness is then isomorphic to the proof of Lemma B.3, since $f_0(w^*)$ is a positive constant multiple of $f_+(w^*)$ (since $w^0 > 0$ in Lemma B.3).

Next, it is straightforward to verify that $j(0) = 0$ and $j(u/r) = 0$ and that $j(w^*) > 0$ for all $w^* \in (0, u/r)$, and since $j(w^*)$ is single-peaked, j attains its maximum on $[0, u/r]$ at some unique $w^*_{PC} \in (0, u/r)$. Finally, it is clear from inspection that $j(w^*) < 1$ for all $w^* \in (0, u/r)$, so in particular, $j(w^*_{PC}) < 1$. $\square$

Lemma B.7. *There exists* $\hat{Q} \in (0, 1)$ *such that if* $\bar{Q} \in (\hat{Q}, 1)$, *then (i) the feasible set lies strictly below the 45-degree line and (ii) there exists a differentiable function* $\bar{w}^0 : w^* \mapsto \bar{w}^0(w^*)$ *such that the feasible set is* $\{(w^*, w^0) \in R_+^2 : w^0 \in (0, \bar{w}^0(w^*)]\}$.

Proof. The proof of the first part of the lemma is by construction. Consider the function

$j : [0, u/r] \to [0, 1]$ defined (for this proof) by

$$j(w) := \begin{cases} Q(u, w, w) & \text{if } w \in (0, u/r] \\ 0 & \text{otherwise.} \end{cases}$$

Now j is continuous for all $w \in (0, u/r]$, and from the proof of Lemma B.4, $\lim_w^0 \to Q(u, w^0, w^0) = 0$, so j is also continuous at 0. Since the domain $[0, u/r]$ is compact, j attains its maximum value at some $\hat{w} \in [0, u/r]$. Since $j(w) > 0$ for all $w > 0$, $\hat{w} \neq 0$. In addition, $j(w) < 1$ for all $w > 0$, so $j(\hat{w}) < 1$. Let $\hat{Q} := j(\hat{w}) < 1$. Then for all $w \in [0, u/r]$, $j(w) \leq \hat{Q}$ and thus for any $\bar{Q} \in (\hat{Q}, 1)$, no point along the 45-degree line is feasible, and since $w^0 \leq w^*$ by definition, it must be that the entire feasible set lies strictly below the 45-degree line.

The existence of the function $\bar{w}^0$ follows from the fact that Q is decreasing in w^0 (Lemma B.1). To establish differentiability, recall that since $\bar{Q} > \hat{Q}$, the feasibility constraint binds and not the constraint $w^0 \leq w^*$; that is, $Q(u, \bar{w^0}(w^*), w^*) = \bar{Q}$. Since Q is continuously differentiable whenever $w^0 \in (0, w^*)$, by the implicit function theorem, $\bar{w}^0(w^*)$ is continuously differentiable whenever $\bar{w}^0(w^*) > 0$. □

C Proofs for Section 5

Proof of Proposition 5.1. As argued in the main text, there is no solution to the maximization problem since w^0 is restricted to be positive. Since Q is decreasing in w^0 and w^*_{PC} maximizes $j(w)$ (where $j(w)$ is defined above Lemma B.6), we have $V(u, w^0, w^*) \leq j(w^*_{PC})$ for all feasible (u, w^0, w^*). Next, note that $j(w^*_{PC})$ is the supremum by continuity. Finally, since w^*_{PC} is the unique maximizer of $j(w)$, it follows that if a sequence of feasible platforms does not converge, the principal's value does not converge, and if the sequence converges to something other than $(0, w^*_{PC}$, the principal's value converges to something less than $j(w^*_{PC})$. □

Proof of Proposition 5.2. Let w^0_{EF} be the maximum w^0 such that (u, w^0, w^*) is feasible for some w^*. Since Q is single-peaked in w^* for each fixed w^0 (Lemma B.3), there is a unique w^*_{EF} such that (u, w^0_{EF}, w^*_{EF}) is feasible, and this is the unique platform which maximizes entry fees. □

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